Quantum Decision Theory

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Are Decision Problems Made of Green Cheese?

Is **decision theory** invariant to the physical environment in which a decision is made?

This seems to be the conventional view.

Likewise, the old view in **computer science** was that the theory of computing could be developed without attention to the particular physical components (silicon, copper, etc.) from which computers are built.

“Computers might as well be made of green cheese.” *

The advent of **quantum computing** showed that the conventional view in computer science was wrong (algorithms: Deutsch-Jozsa 1992; Grover 1996; Shor 1997).

We will argue that the availability of quantum information resources means that the conventional view in decision theory is also wrong.

* Our thanks to Samson Abramsky, who attributes this aphorism to his Ph.D. advisor.

Quantum vs. Classical Signals

We first examine a **classical baseline**:

What happens when a decision maker (DM) has access to classical signals and can make his choices contingent on the realization of those signals?

We then ask:

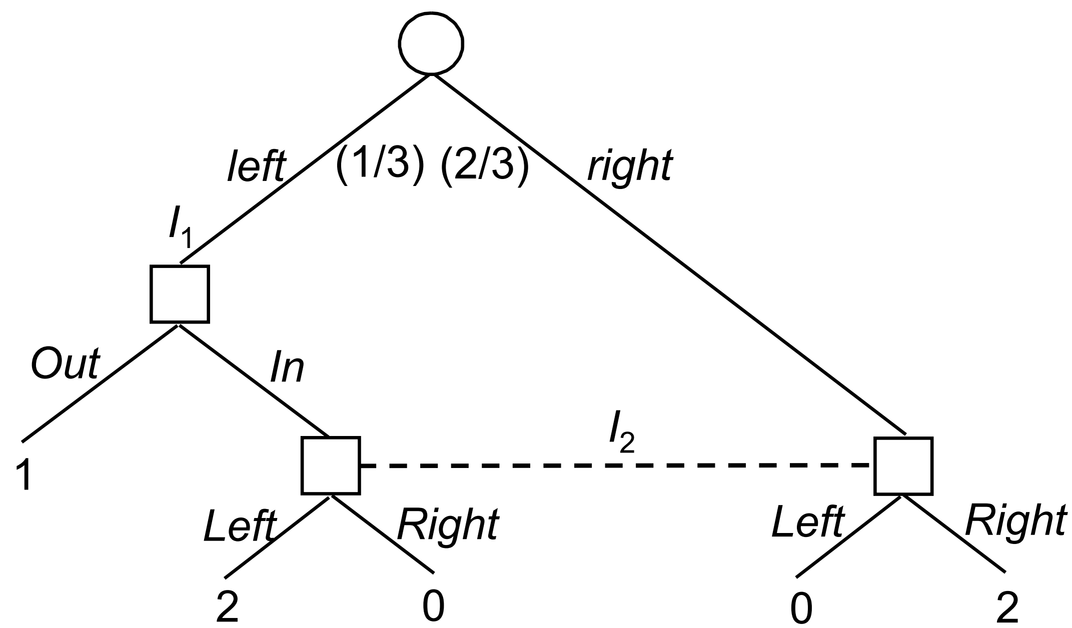
Does giving the DM access to **quantum**, not just classical, signals, lead to an improvement in what he can achieve?

We can interpret the addition of signals --- classical or quantum --- to a decision problem in two ways:

i. Signals represent an **extra resource** which a DM might be able to employ.

ii. Signals are **omnipresent** in the environment, and this is simply the correct analysis of decision making.

Example #1



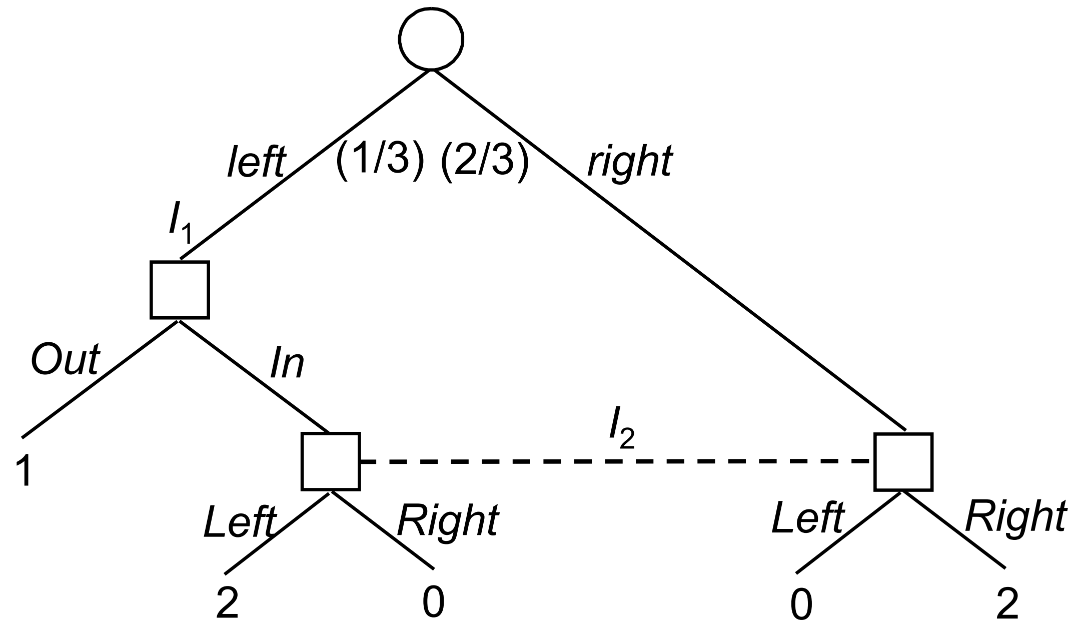
The circular node belongs to Nature, and the square nodes belong to the DM.

This is a decision tree with **imperfect recall** (Kuhn 1950, 1953).

At information set I_2, the DM does not remember his previous choice (if any).

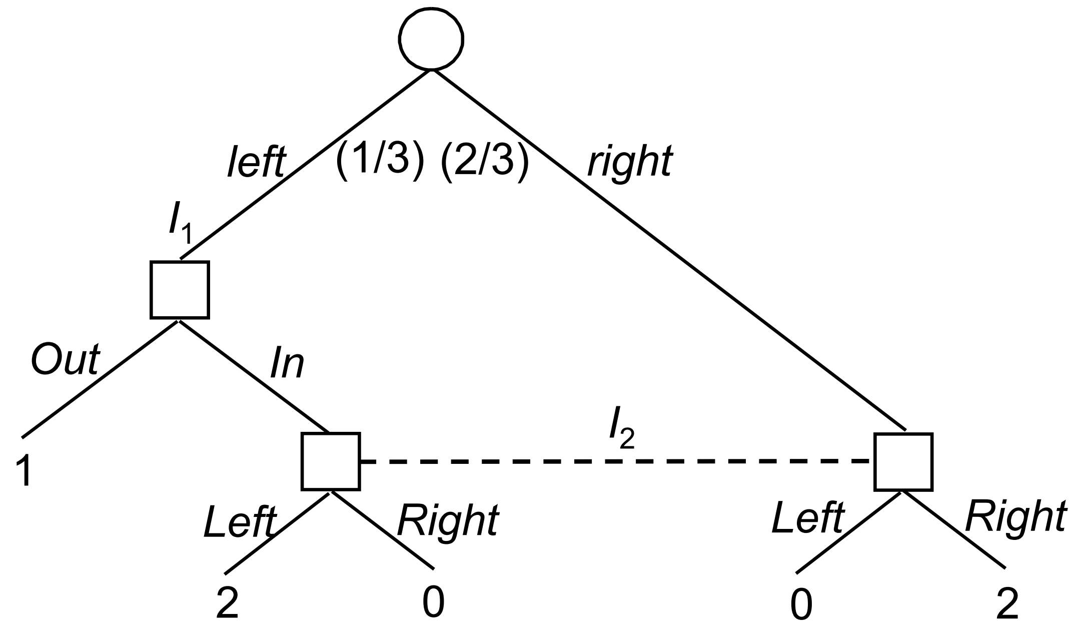
(Obviously, this is not yet a formal definition of perfect recall.)

Highest Expected Payoff



In-Left	2/3
In-Right	4/3
Out-Left	1/3
Out-Right	5/3

Can Signals Help?

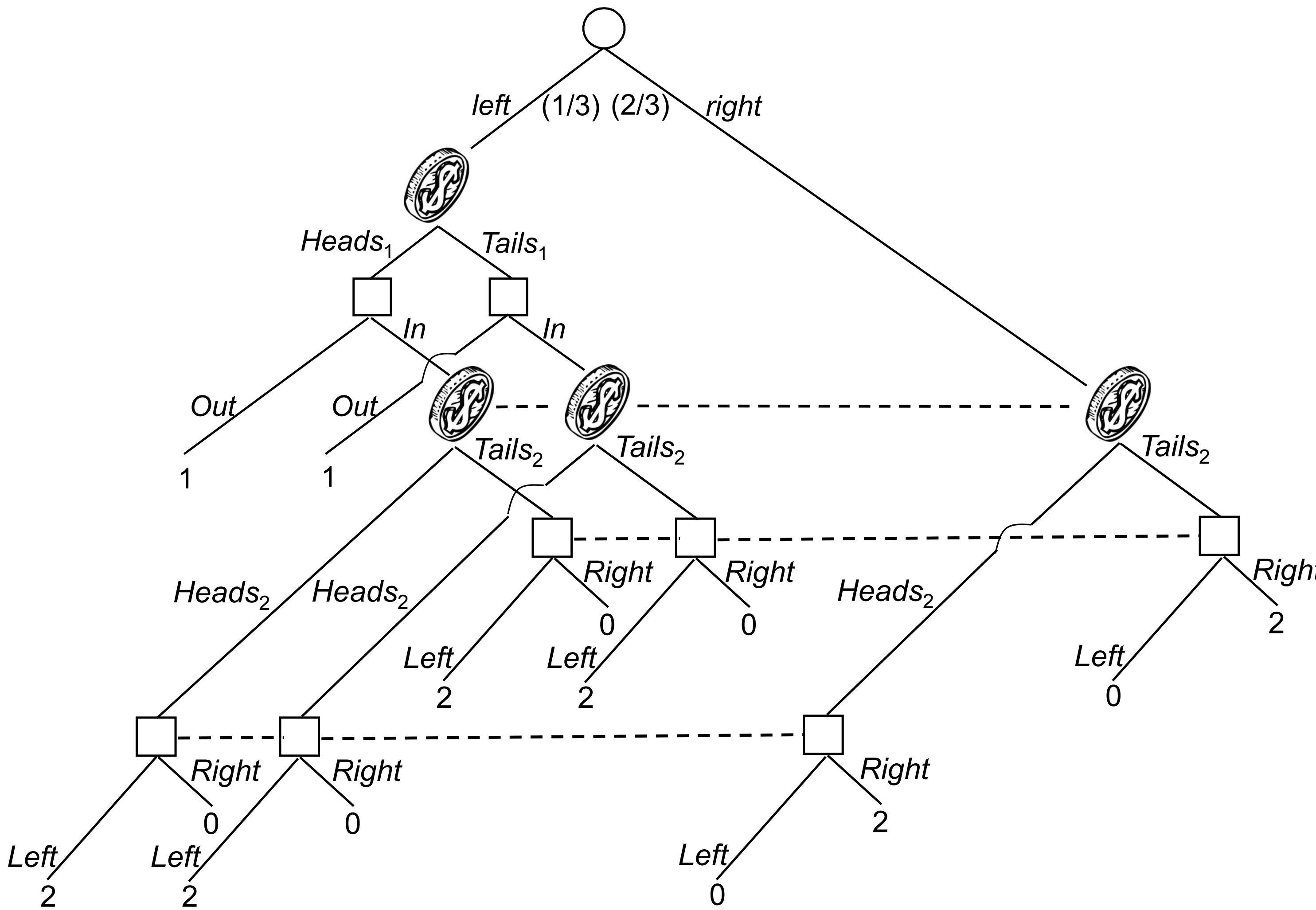


Perhaps, signals could carry information through the tree which the DM is unable to carry himself.

Signals might make up for a **lack** of memory.

Could they even be a **constituent** of memory?

Adding Signals



Signal Structures

Each possible path through the tree crosses certain information sets of the DM in a certain order.

$\Pr_{I_1}$

H_1	T_1
α	β

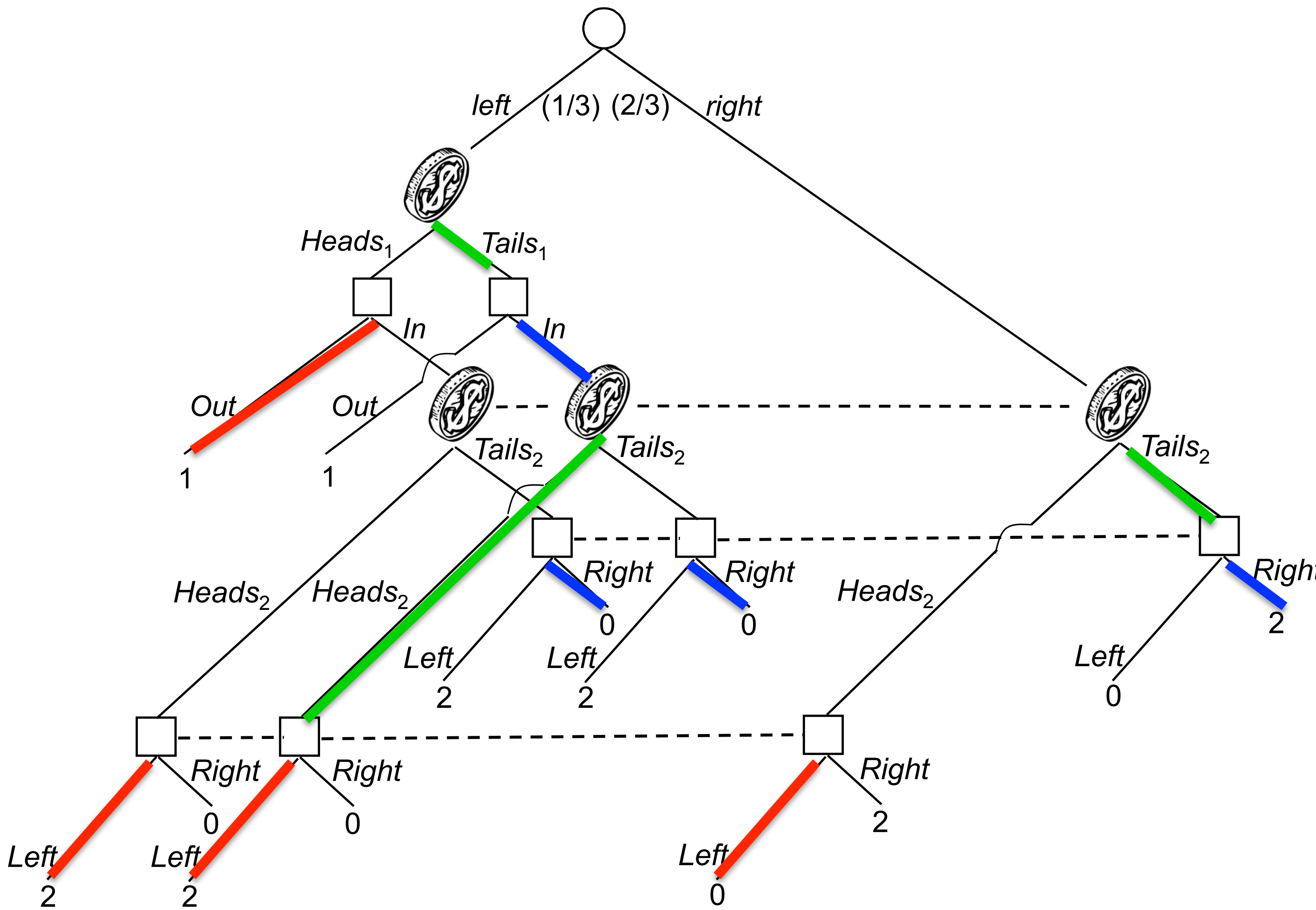
$\Pr_{I_2}$

H_2	T_2
γ	δ

$\Pr_{I_1 I_2}$

	H_2	T_2
H_1	ε	ζ
T_1	η	θ

Highest Expected Payoff in the Augmented Tree



Set $\delta = \eta = 1$. The expected payoff is 2 > 5/3!

The No Signaling Condition

But, the behavior of the second coin is affected by the toss of the first coin.

In this sense, information is carried between information sets, so that it is not surprising that there is an improvement.

The **No Signaling Condition**:

> Consider two tuples of information sets and the two associated signal probability measures. The marginals of these two measures --- with respect to common sub-tuples --- must agree.

(The terminology is from quantum mechanics, and can be a bit confusing in decision theory.)

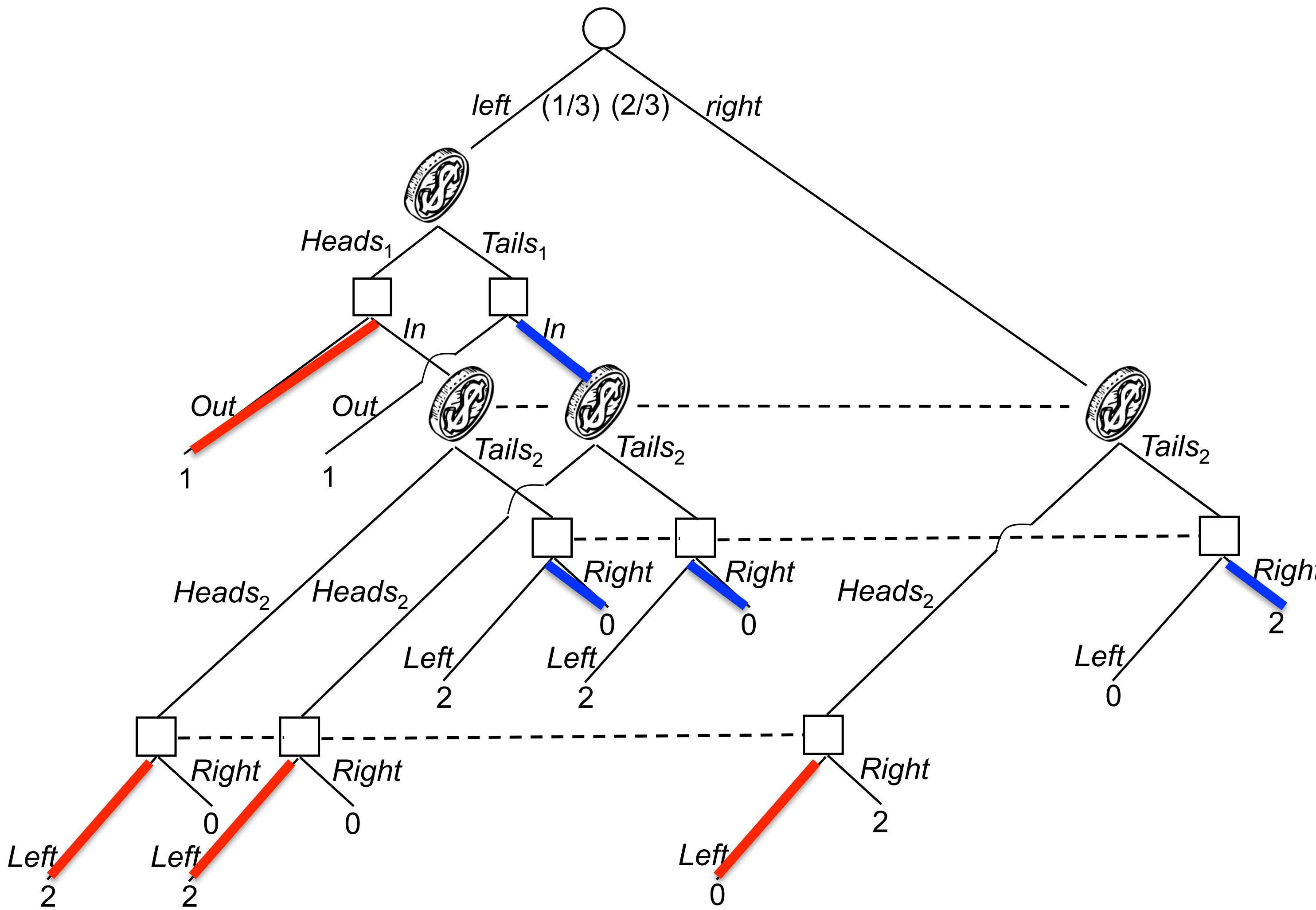
In the previous example, the condition implies:

$$\alpha = \varepsilon + \zeta$$

$$\gamma = \varepsilon + \eta$$

which rules out $\delta = \eta = 1$.

No Signaling in the Augmented Tree



Expected payoff from the strategy shown:

$$\varepsilon\times(\tfrac{1}{3}\times 1 + \tfrac{2}{3}\times 0) + \zeta\times(\tfrac{1}{3}\times 1 + \tfrac{2}{3}\times 2) + \eta\times(\tfrac{1}{3}\times 2 + \tfrac{2}{3}\times 0) + \theta\times(\tfrac{1}{3}\times 0 + \tfrac{2}{3}\times 2)$$

A Conjecture

Conjecture:

Fix a Kuhn tree. The highest expected payoff to a DM in an augmented tree with signals satisfying **No Signaling** is the same as that in the tree without signals.

This is false, as we shall see!

But we do have:

Proposition:

Fix a Kuhn tree. The highest expected payoff to a DM in an augmented tree with **classical** signals is the same as that in the tree without signals. Moreover, No Signaling will be satisfied.

Of course, we have to say what we mean by “classical”.

Classicality

The **Classicality Condition**:

> Let $\{I_1, I_2, \ldots\}$ be the set of information sets for the DM. There is a probability measure μ on the product, over all $I_1, I_2, \ldots$, of the associated signal sets, such that: For each information tuple $I_{i_1} I_{i_2} \ldots$ that arises in the tree, the probability measure $\Pr_{I_{i_1} I_{i_2} \ldots}$ is obtained from μ by marginalization.

In short, there is a joint state space!

Note: This condition is well-defined since, in a Kuhn tree, each path crosses a given information set at most once.

Implications of Classicality

Proposition:

Classicality implies No Signaling.

Proof:

Immediate by the properties of marginals.

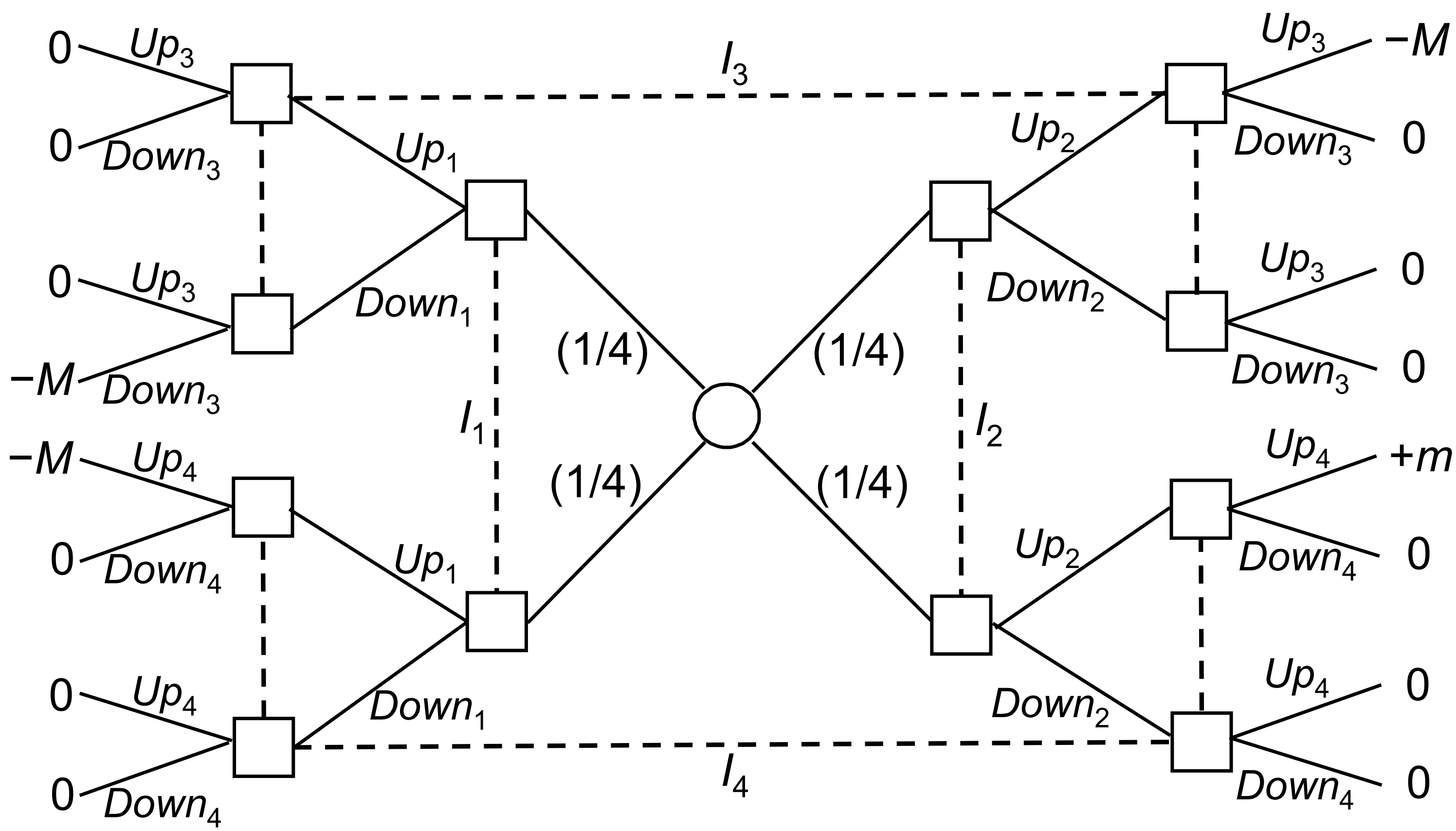
Proposition:

Fix a Kuhn tree. The highest expected payoff a DM can achieve with signals satisfying Classicality is the same as that without signals.

Proof:

Under Classicality, we can write the expected payoff to a strategy in the augmented tree as a convex combination of expected payoffs to strategies in the original tree.

Example #2



Assume $0 < m < M$.

The DM's expected payoff with classical signals is at most 0.

Proof: Analyze without signals and then appeal to the previous proposition.

A Signal Structure

$\Pr_{I_1 I_3}$

	H_3	T_3
H_1	Φ^3	Φ^2
T_1	Φ^2	0

$\Pr_{I_2 I_3}$

	H_3	T_3
H_2	0	Φ^3
T_2	Φ	Φ^4

$\Pr_{I_1 I_4}$

	H_4	T_4
H_1	0	Φ
T_1	Φ^3	Φ^4

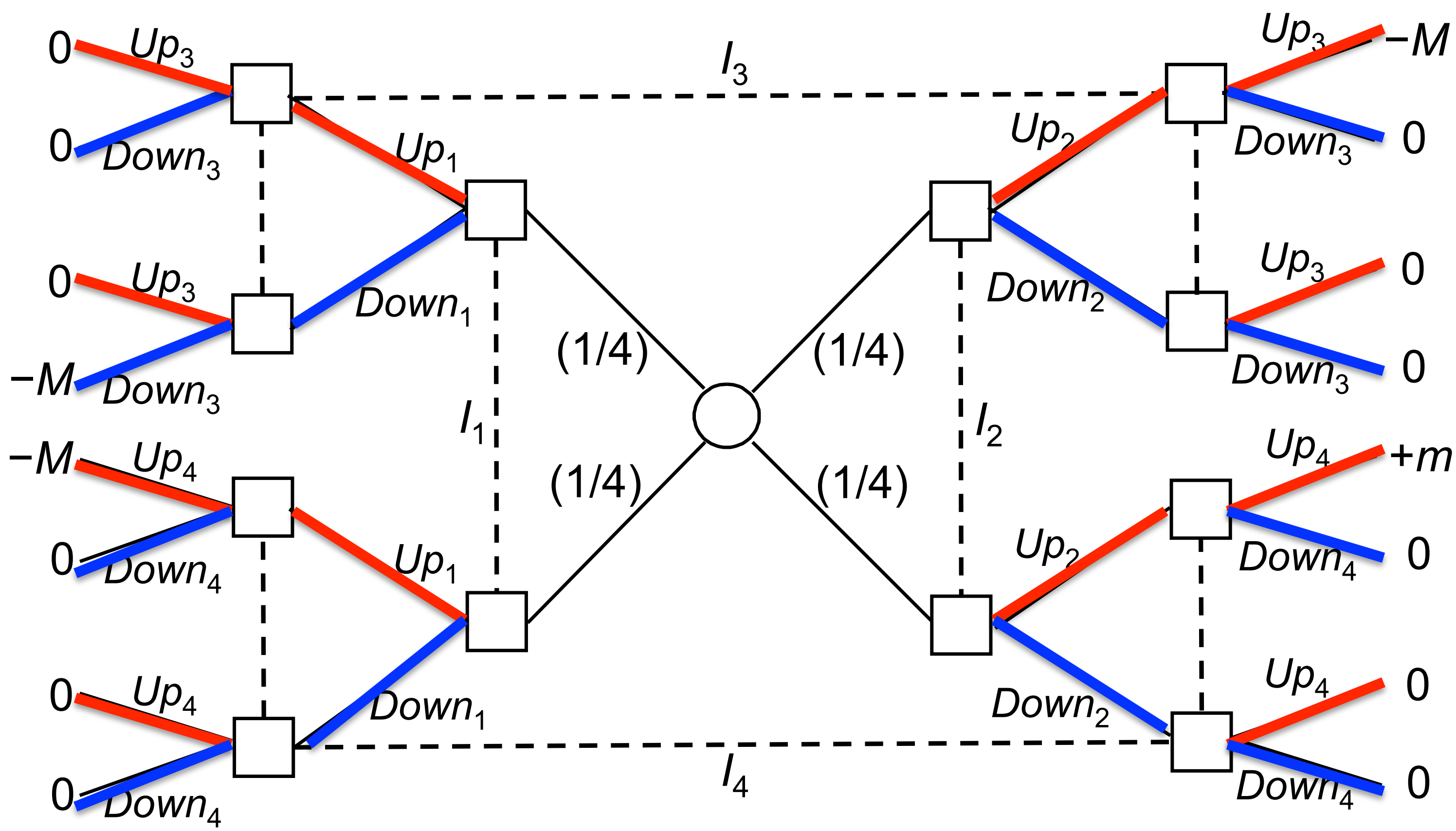
$\Pr_{I_2 I_4}$

	H_4	T_4
H_2	Φ^5	Φ^4
T_2	Φ^4	Φ

$\Phi = \dfrac{2}{1+\sqrt{5}}$ is the inverse of the Golden Ratio.

No Signaling is satisfied (use $\Phi^2 + \Phi = 1$).

Augmented Tree



The expected payoff is $\frac{1}{4} \times \Phi^5 \times m > 0!$

Physical Realization

This signal structure can be physically realized in a quantum-mechanical (QM) system (Hardy 1993).

The signal (**Heads** or **Tails**) at I_1 is the outcome obtained from performing a certain measurement on particle #1.

The signal (**Heads** or **Tails**) at I_2 is the outcome obtained from performing a different measurement on particle #1.

The signal (**Heads** or **Tails**) at I_3 is the outcome obtained from performing a certain measurement on particle #2.

The signal (**Heads** or **Tails**) at I_4 is the outcome obtained from performing a different measurement on particle #2.

The key is that the two particles are **entangled** in a particular way.

A Joint State Space?

	I_1	I_2	I_3	I_4
ω^0	H	H	H	H
ω^1	H	H	H	T
ω^2	H	H	T	H
ω^3	H	H	T	T
ω^4	H	T	H	H
ω^5	H	T	H	T
ω^6	H	T	T	H
ω^7	H	T	T	T
ω^8	T	H	H	H
ω^9	T	H	H	T
ω^{10}	T	H	T	H
ω^{11}	T	H	T	T
ω^{12}	T	T	H	H
ω^{13}	T	T	H	T
ω^{14}	T	T	T	H
ω^{15}	T	T	T	T

Non-Classicality of the Signal Structure

We know from our proposition about classicality that the signal structure cannot arise from a joint state space.

Let us also give a direct proof. We try:

$\mu(\omega^{10}) + \mu(\omega^{11}) + \mu(\omega^{14}) + \mu(\omega^{15}) = 0,$

$\mu(\omega^{0}) + \mu(\omega^{1}) + \mu(\omega^{8}) + \mu(\omega^{9}) = 0,$

$\mu(\omega^{0}) + \mu(\omega^{2}) + \mu(\omega^{4}) + \mu(\omega^{6}) = 0,$

but then find this contradicts:

$\mu(\omega^{0}) + \mu(\omega^{2}) + \mu(\omega^{8}) + \mu(\omega^{10}) > 0!$

Discussion

QM says that the two measurements on particle #1 (resp. particle #2) cannot have jointly well-defined outcomes.

This is a statement of the **incompatibility** or **non-commutativity** of various observables in QM (most famously: position and momentum).

It is the physical reason why there is no joint state space.

So this is a striking case of how a ‘weakness’ (incompatibility) becomes a ‘strength’ (entanglement).

Related analyses:

> Is there is a **local hidden-variable model** that induces the empirical outcome probabilities (Bell 1964)?
>
> Is there a joint state space with a **signed probability measure** that induces the empirical outcome probabilities (Abramsky and Brandenburger, *New Journal of Physics*, 2011)? (Of course, all empirical probabilities must be non-negative.)

Entanglement in Living Systems

The dawn of quantum biology

The key to practical quantum computing and high-efficiency solar cells may lie in the messy green world outside the physics lab.

BY PHILIP BALL

DARREN HOPKINS

On the face of it, quantum effects and living organisms seem to occupy utterly different realms. The former are usually observed only on the nanometre scale, surrounded by hard vacuum, ultra-low temperatures and a tightly controlled laboratory environment. The latter inhabit a macroscopic world that is warm, messy and anything but controlled. A quantum phenomenon such as 'coherence', in which the wave patterns of every part of a system stay in step, wouldn't last a microsecond in the tumultuous realm of the cell.

Or so everyone thought. But discoveries in recent years suggest that nature knows a few tricks that physicists don't: coherent quantum processes may well be ubiquitous in the natural world. Known or suspected examples range from the ability of birds to navigate using Earth's magnetic field to the inner workings of photosynthesis — the process by

Summary

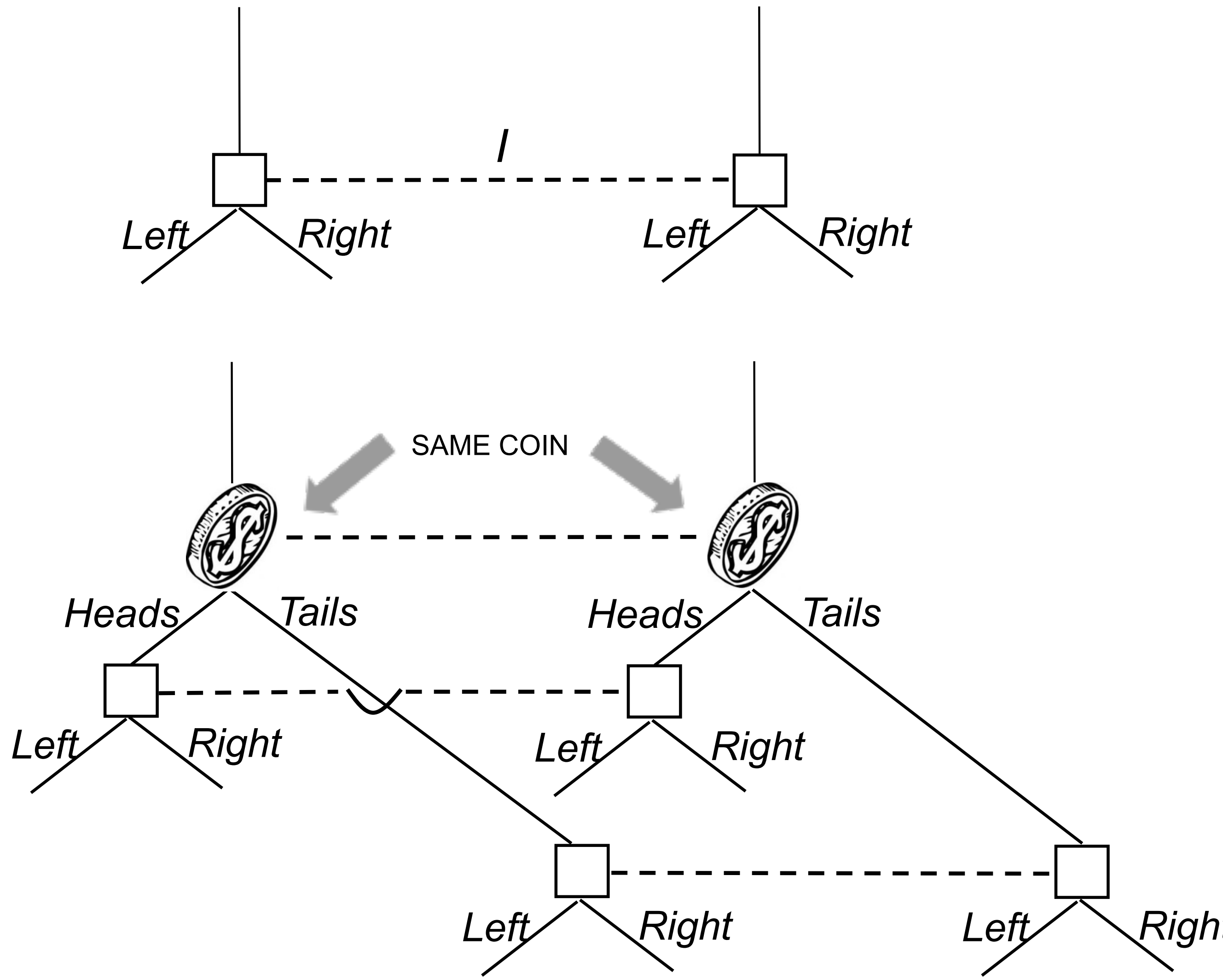
Type of signal:

Type of tree:

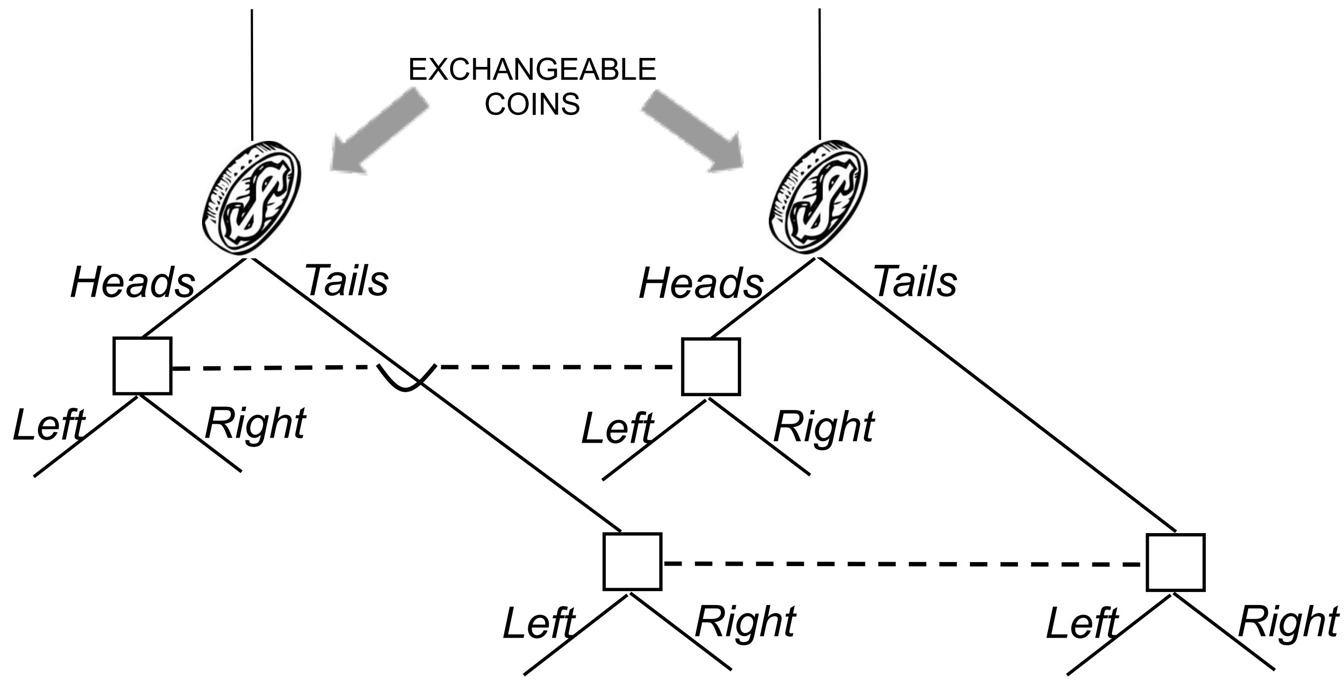
	None	Classical	Quantum
Perfect-recall Kuhn	**0**	**0**	**0**
Imperfect-recall Kuhn	**0**	**0**	**+**
Non-Kuhn	**0**	**+**	**++**

The Formulation So Far

The formulation we have used so far:



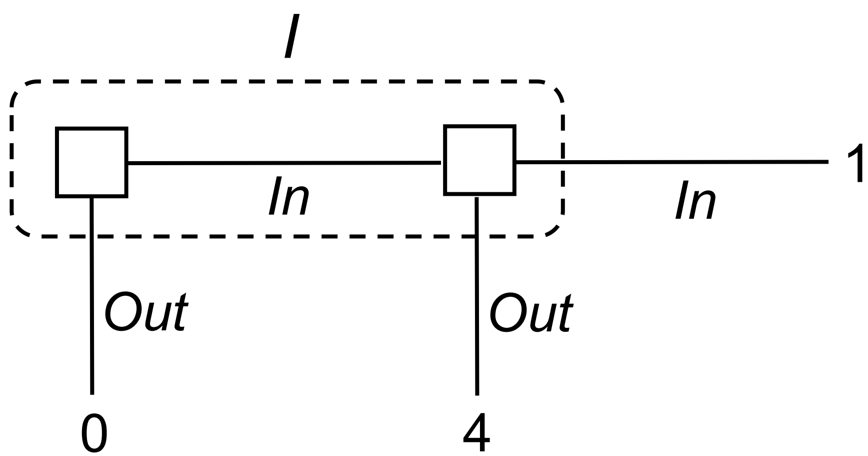
A Second Formulation



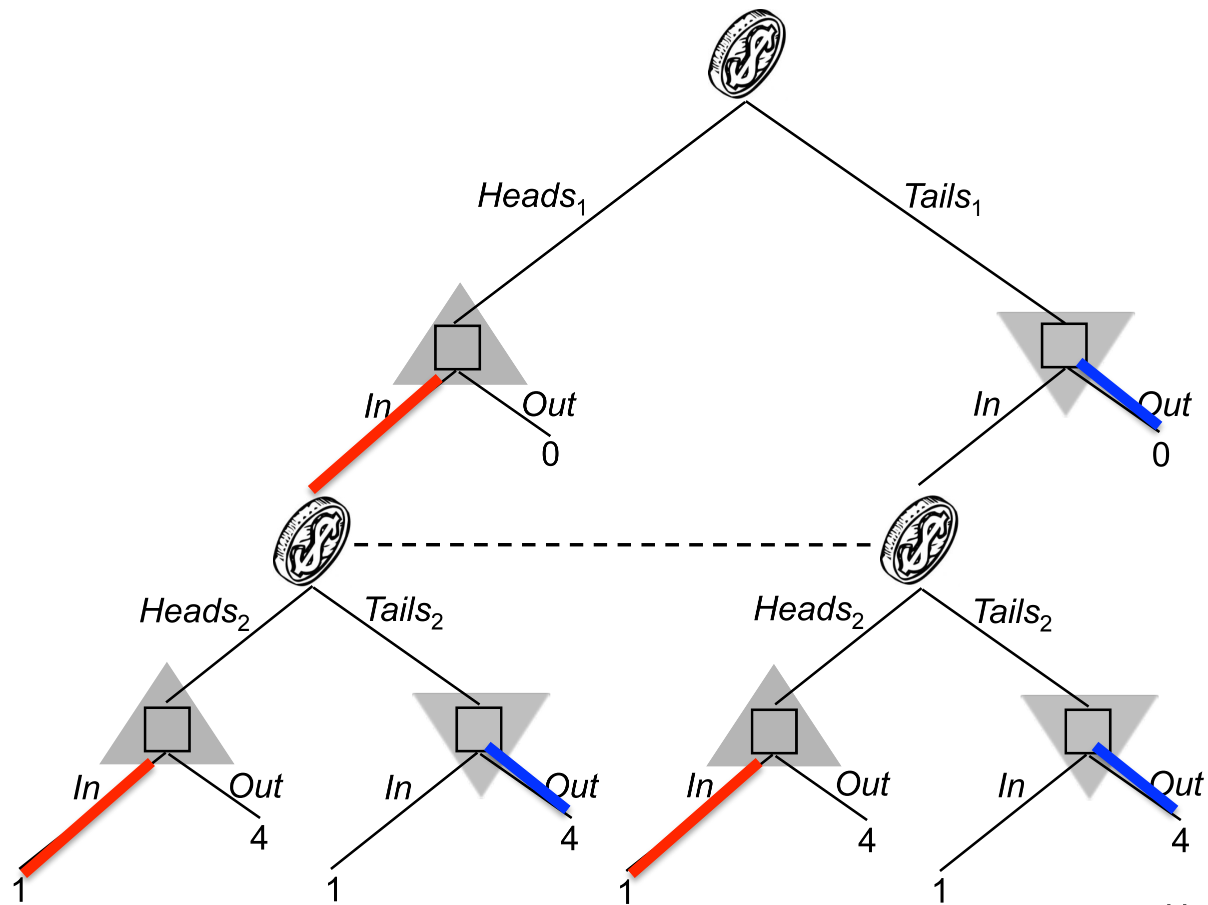
Non-Kuhn Trees

The second formulation leaves unchanged our results so far.

But it can make a difference in **non-Kuhn trees** (Isbell 1957, Piccione and Rubinstein 1997):



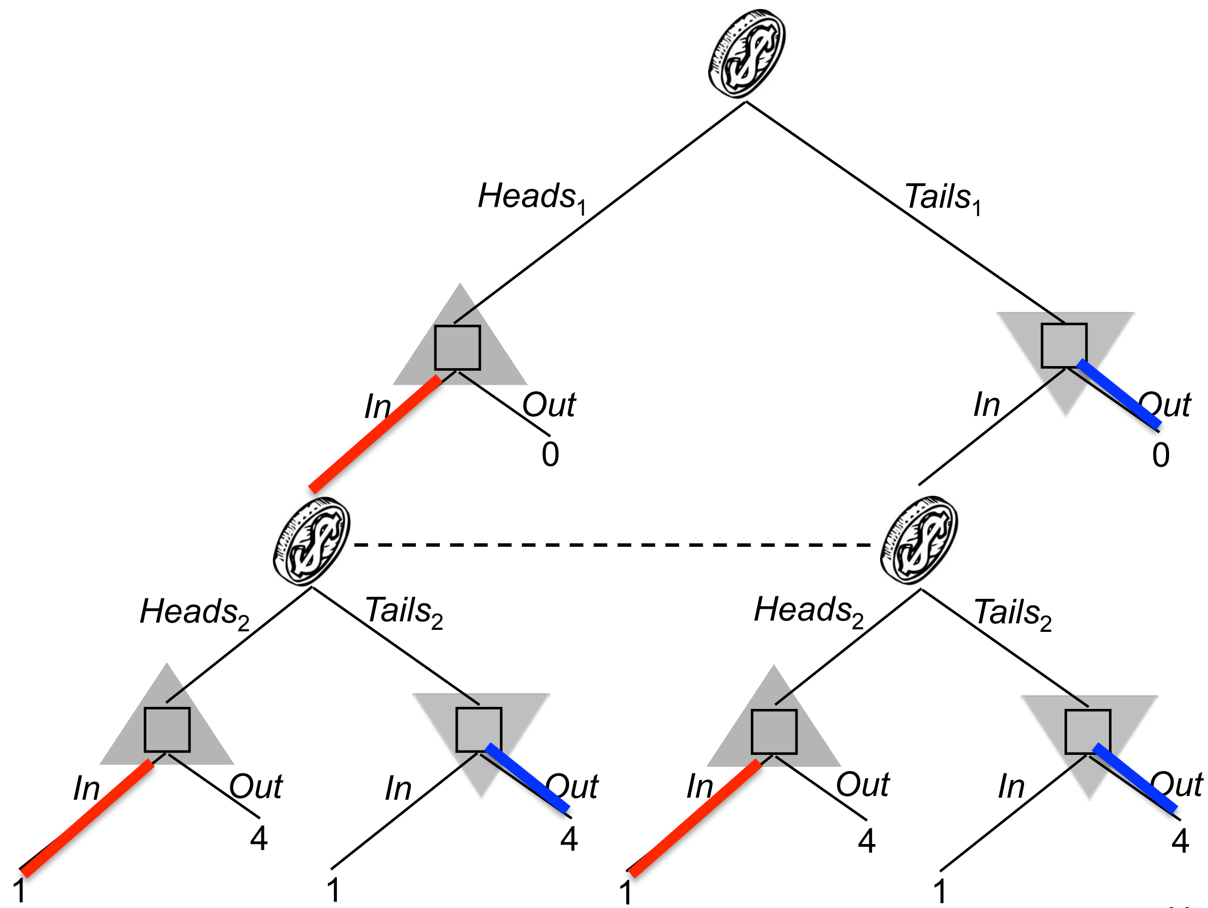
Adding i.i.d. Signals



The expected payoff is 5/4 > 1! (Isbell 1957)

	$Heads_2$	$Tails_2$
$Heads_1$	1/4	1/4
$Tails_1$	1/4	1/4

Adding Exchangeable Signals



The expected payoff is 2 > 5/4!

	Heads$_2$	Tails$_2$
Heads$_1$	0	1/2
Tails$_1$	1/2	0

Summary Again

Type of signal:

Type of tree:

	None	Classical	Quantum
Perfect-recall Kuhn	**0**	**0**	**0**
Imperfect-recall Kuhn	**0**	**0**	**+**
Non-Kuhn	**0**	**+**	**++**